Lecture 5
09.03.2020

Scribe: Abdulrahman Dandaşi - Gürol Sağlam
Lecturer: Özlem Salehi Köken

5.4 Quantum Teleportation

Alice wants to send the state of her qubit to Bob but she has access to only classical channel.

We will assume that Alice and Bob share an entangled pair of qubits. In that case quantum state of Alice can be sent using a classical channel sending only two classical bits.

Alice and Bob initially share the Bell state $|B_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Alice has the first and Bob has the second qubit. Suppose that Alice wants to send state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. The state of the overall three qubits is given by

$$\begin{aligned}|\phi\rangle|B_{00}\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)\end{aligned}$$

where the first qubit is the qubit to be sent, and the last two qubits are the entangled pair.

- Alice applies CNOT to her own qubits where the qubit she wants to send is the control.

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

- Alice applies H to the qubit she wants to send (first qubit).

$$\frac{1}{\sqrt{2}}\left(\frac{(\alpha|000\rangle + \alpha|100\rangle)}{\sqrt{2}} + \frac{(\alpha|011\rangle + \alpha|111\rangle)}{\sqrt{2}} + \frac{(\beta|010\rangle - \beta|110\rangle)}{\sqrt{2}} + \frac{(\beta|001\rangle - \beta|101\rangle)}{\sqrt{2}}\right)$$

$$= \frac{1}{2}(|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle))+\frac{1}{2}(|01\rangle\otimes(\beta|0\rangle+\alpha|1\rangle))+\frac{1}{2}(|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle))+\frac{1}{2}(|11\rangle\otimes(-\beta|0\rangle+\alpha|1\rangle))$$

- Alice measures her own qubits.
- Alice sends the measurement outcome to Bob - two bits of classical information.
- Depending on the measurement outcome, Bob either applies I, X, Z or X and Z to his qubit.

Measurement of Alice	State of Bob's qubit	Operation by Bob
$\frac{1}{4}\|00\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	I
$\frac{1}{4}\|01\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	X
$\frac{1}{4}\|10\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$	Z
$\frac{1}{4}\|11\rangle$	$-\beta\|0\rangle + \alpha\|1\rangle$	first X, then Z

At the end of the protocol, Bob's qubit is now exactly in the state $\alpha|0\rangle + \beta|1\rangle$, whereas Alice's qubit is destroyed. Copying a qubit is not possible due to No Cloning Theorem.

Note that this is not faster than light speed as we also need classical communication.

6 Introductory Quantum Algorithms

In this section we will be talking about some basic quantum algorithms.

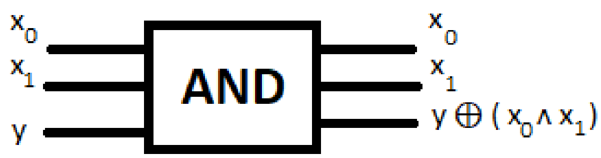
6.1 Simulating Classical Gates and Algorithms

Any irreversible computation can be transformed into a reversible computation. We construct a circuit with three inputs and three outputs, so that input is a part of the output as well, so that the computation is reversible.

Example:

AND

in	in	out
0	0	0
0	1	0
1	0	0
1	1	1



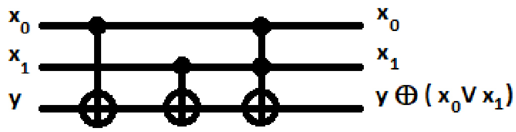
We construct the following table setting $y = 0$ and see that we can use Toffoli (CCX) gate to implement AND gate in a reversible manner.

in	in	out	in	in	out
0	0	0	0	0	0
0	1	0	0	1	0
1	0	0	1	0	0
1	1	0	1	1	1

Example: Let's implement OR gate using CNOTs and CCNOTs.

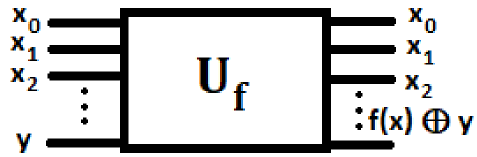
OR

in	in	out
0	0	0
0	1	1
1	0	1
1	1	1



in	in	out	in	in	out
0	0	0	0	0	0
0	1	0	0	1	1
1	0	0	1	0	1
1	1	0	1	1	1

Consider any arbitrary function $f\{0,1\}^m \rightarrow \{0,1\}$. Such a function can be implemented by a unitary operator as follows.



$$U_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$$

6.2 Phase Kickback

Consider a unitary operator $U_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$. Let's set $|y\rangle$ to be $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

$$U_f : \left(|x\rangle|\frac{|0\rangle - |1\rangle}{\sqrt{2}}\rangle\right) \rightarrow \left(\frac{U_f(|x\rangle|0\rangle) - U_f(|x\rangle|1\rangle}{\sqrt{2}}\right) = \frac{|x\rangle|f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}}$$

Consider the two cases:

- f(x) = 0

$$\frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}} = |x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

- f(x) = 1

$$\frac{|x\rangle|1\rangle - |x\rangle|0\rangle}{\sqrt{2}} = |x\rangle\left(\frac{|1\rangle - |0\rangle}{\sqrt{2}}\right)$$

We can write it as

$$U_f : |x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle.$$

This is called phase kickback, as the phase -1 is kicked back in front of the first register when $f(x) = 1$. In fact, when $f(x) = 1$, its effect is applying a NOT gate and the state $|-\rangle$ is an eigenstate of the NOT operator with eigenvalue -1, that is, NOT$|-\rangle$ = -$|-\rangle$.

6.3 Deutsch Algorithm

Problem: Suppose we are given a function $f\{0,1\} \to \{0,1\}$. We say that f is

- Constant if $f(0) = f(1)$,
- Balanced if $f(0) \neq f(1)$.

The aim is to understand whether f is constant or balanced by making queries to f. Note that we treat f as a black box or an oracle. That is, we make queries to f but we cannot look inside it.

Classically, we need to make 2 queries to decide if f is constant or balanced. Now we will see that, Deutsch algorithm makes only 1 queries to decide whether f is constant or balanced.

To start with note that

- If f is constant, then $f(0) \oplus f(1) = 0$
- If f is balanced, then $f(0) \oplus f(1) = 1$

Suppose that we are given the following operator $U_f|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$.

<u>1^{st} Try:</u> Initially, let $|x\rangle = 0$ and $|y\rangle = 0$, then apply H to $|x\rangle$:

$$\begin{aligned}|00\rangle &\to \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\end{aligned}$$

Now apply U_f:

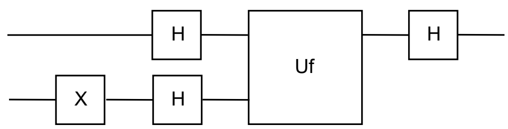
$$\begin{aligned}&U_f\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) \\ &= \frac{1}{\sqrt{2}}U_f|00\rangle + \frac{1}{\sqrt{2}}U_f|10\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle|0 \oplus f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|0 \oplus f(1)\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle\end{aligned}$$

We computed $f(0)$ and $f(1)$ in superposition. But it is not enough since if we make an observation we either observe $|0\rangle|f(0)\rangle$ or $|1\rangle|f(1)\rangle$.

Algorithm:

- Apply H to the first qubit (input).
- Set the second qubit to state $|-\rangle$.
- Apply U_f.
- Apply H to the first qubit again.

Implementation:



Analysis:

We start in

$$|\psi\rangle = |0\rangle|-\rangle.$$

Apply H to the first qubit:

$$|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|-\rangle$$

Apply U_f:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle|-\rangle + \frac{1}{\sqrt{2}}|1\rangle|-\rangle$$

$$|\psi_2\rangle = \frac{(-1)^{f(0)}|0\rangle|-\rangle}{\sqrt{2}} + \frac{(-1)^{f(1)}|1\rangle|-\rangle}{\sqrt{2}}$$

Check that:

$$(-1)^{f(0)}(-1)^{f(1)} = (-1)^{f(0)\oplus f(1)}$$

$$|\psi_2\rangle = (-1)^{f(0)}\left(\frac{|0\rangle + |-1^{f(0)\oplus f(1)}\rangle}{\sqrt{2}}|1\rangle\right)|-\rangle$$ [1]

[1] $(-1)^{f(0)} \cdot (-1)^{f(0)} \cdot (-1)^{f(1)} = (-1)^{f(1)}$

- If f is constant, $f(0) \oplus f(1) = 0$

$$|\psi_2\rangle = (-1)^{f(0)} \frac{|0\rangle + |1\rangle}{\sqrt{2}} |-\rangle$$

- If f is balanced, $f(0) \oplus f(1) = 1$

$$|\psi_2\rangle = (-1)^{f(0)} \frac{|0\rangle - |1\rangle}{\sqrt{2}} |-\rangle$$

Apply H to first qubit, if f is constant we get:

$$|\psi_3\rangle = (-1)^{f(0)} |0\rangle |-\rangle$$

If f is balanced we get:

$$|\psi_3\rangle = (-1)^{f(0)} |1\rangle |-\rangle$$

Measure the first qubit. If it is 0, then f is constant, and if it is 1, then f is balanced. We determine whether f is constant or balanced with probability 1 by making only a single query.