Özyeğin University **Spring 2020**

CS480/CS580 Quantum Computing

Lecture 4 - Quantum Circuit Model, Entanglement, Superdense Coding

02.03.2020

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4 Quantum Circuit Model

In this section, we will be talking about the quantum circuit model.

4.1 1-Qubit Gates

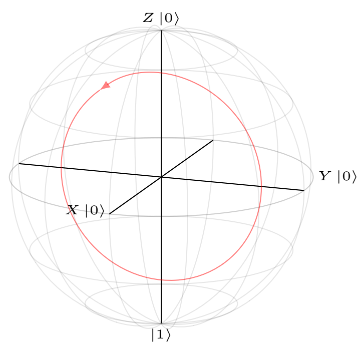
Not (X) : Corresponds to a 180 rotation around X - Axis

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle$$



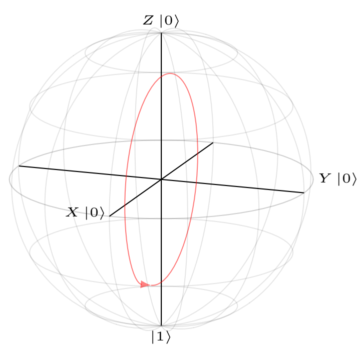
Y-Gate (Y) : Corresponds to a 180 rotation around Y - Axis

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = i|1\rangle$$

$$Y|1\rangle = -i|0\rangle$$

$$Y(\alpha|0\rangle + \beta|1\rangle) = -\beta i|0\rangle + \alpha|1\rangle$$



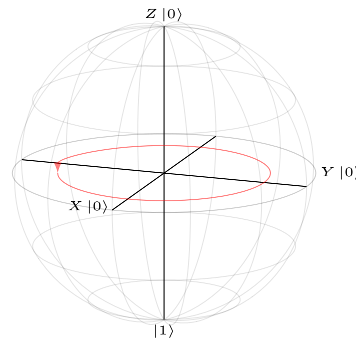
Z-Gate (Z) : Corresponds to a 180 rotation around Z - Axis

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z\left|0\right\rangle = \left|0\right\rangle$$

$$Z\left|1\right\rangle = -\left|1\right\rangle$$

$$Z(\alpha\left|0\right\rangle + \beta\left|1\right\rangle) = \alpha\left|0\right\rangle + -\beta\left|1\right\rangle$$



Rotation Gates

Let $|\Psi\rangle$ be a quantum state and let μ be a unitary operator. The action of μ on $|\Psi\rangle$ can be thought as a rotation on the Bloch sphere

$$R_x(\theta) = e^{-i\theta X/2}$$

$$R_y(\theta) = e^{-i\theta Y/2}$$

$$R_z(\theta) = e^{-i\theta Z/2}$$

If $A^2 = I, e^{iAx} = cos(x)I + isin(x)A$

$$R_x(\theta) = cos\left(\frac{\theta}{2}\right)I + isin\left(\frac{\theta}{2}\right)X = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & -isin\left(\frac{\theta}{2}\right) \\ -isin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_y(\theta) = cos\left(\frac{\theta}{2}\right)I + isin\left(\frac{\theta}{2}\right)Y = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & -sin\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta Z/2} = cos\left(\frac{\theta}{2}\right)I - \left(\frac{\theta}{2}\right)Z$$

$$= \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(\frac{\theta}{2}\right) \end{pmatrix} - \begin{pmatrix} isin\left(\frac{\theta}{2}\right) & 0 \\ 0 & -isin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} cos\left(\frac{\theta}{2}\right) - isin\left(\frac{\theta}{2}\right) & 0 \\ 0 & cos\left(\frac{\theta}{2}\right) + isin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

Example:

Let $|\Psi| = cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i\varphi}sin\left(\frac{\sigma}{2}\right)|1\rangle$

$$\begin{aligned}
R_z(\theta) &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} cos\left(\frac{\sigma}{2}\right) \\ e^{i\varphi}sin\left(\frac{\sigma}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\theta/2}cos\left(\frac{\sigma}{2}\right) \\ e^{i\theta/2}e^{i\varphi}sin\left(\frac{\sigma}{2}\right) \end{pmatrix} \\
&= e^{-i\theta/2}cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i\theta/2}e^{i\varphi}sin\left(\frac{\sigma}{2}\right)|1\rangle \\
&= e^{-i\theta/2}(cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i\theta}e^{i\varphi}sin\left(\frac{\sigma}{2}\right)|1\rangle) \\
&= e^{-i\theta/2}(cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i(\theta+\varphi)}sin\left(\frac{\sigma}{2}\right)|1\rangle)
\end{aligned}$$

Effect of $R_z(\theta)$ is to change the angle φ to $\varphi + \theta$ which is a rotation around z-axis

Theorem 1. *Suppose μ is a 1-Qubit unitary gate. Then there exists α, β, γ and δ such that $\mu = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$*

Theorem 2. *Any unitary μ can be expressed as $\mu = e^{i\alpha}AXBXC$ where $ABC = I$, A,B,C are unitary*

S-Gate

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S = R_z(\pi/2)$$

$$S^2 = Z$$

T-Gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}$$

$$T = R_z(\pi/4)$$

$$T^2 = S$$

Hadamard Gate (H)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|+\rangle = H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$
\begin{aligned}
H\,|+\rangle &= H\left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle\right) \\
&= \frac{1}{2}\,|0\rangle + \frac{1}{2}\,|1\rangle + \frac{1}{2}\,|0\rangle - \frac{1}{2}\,|1\rangle \\
&= |0\rangle
\end{aligned}
$$

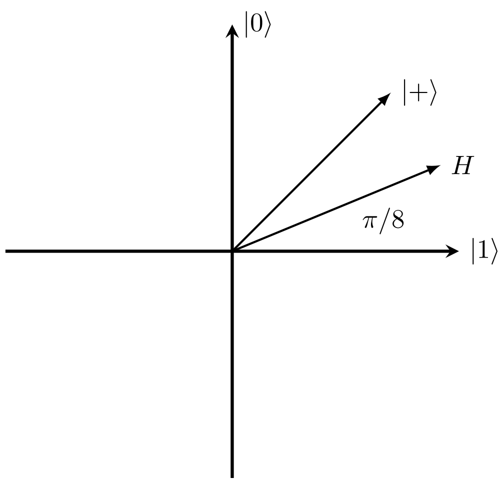
$$H^2 = I$$

$$
\begin{aligned}
(H \otimes H)(|00\rangle) &= H^{\otimes 2}\,|00\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{2}\,|00\rangle + \frac{1}{2}\,|01\rangle + \frac{1}{2}\,|10\rangle + \frac{1}{2}\,|11\rangle
\end{aligned}
$$

$H^{\otimes n}\,|0...0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$ where x is written in binary

$$H^{\otimes 3}\,|000\rangle = \frac{1}{\sqrt{8}} \sum_{x=0}^{7} |x\rangle = \frac{1}{\sqrt{8}}\,|000\rangle + \frac{1}{\sqrt{8}}\,|001\rangle + ... + \frac{1}{\sqrt{8}}\,|111\rangle$$

Hadamard corresponds to a rotation of π around $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. When restricted to real amplitudes, Hadamard is a reflection matrix in 2D plane over angle $\pi/8$



4.2 Multi-Qubit Gates

Controlled Not (CNOT)

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$CNOT\,|10\rangle = |11\rangle$$

$$CNOT\,|11\rangle = |10\rangle$$

$$CNOT\,|01\rangle = |01\rangle$$

$$CNOT\,|00\rangle = |00\rangle$$

Example:

Let $|\Psi\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|10\rangle = \left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\right) \otimes |0\rangle$

$CNOT(|\Psi\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle)$

Controlled Z-Gate (CZ)

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$CZ\,|00\rangle = |00\rangle$$

$$CZ\,|01\rangle = |01\rangle$$

$$CZ\,|10\rangle = |10\rangle$$

$$CZ\,|11\rangle = -\,|11\rangle$$

Toffoli Gate (CCX)

$$CCX = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$CCX\,|000\rangle = |000\rangle$$

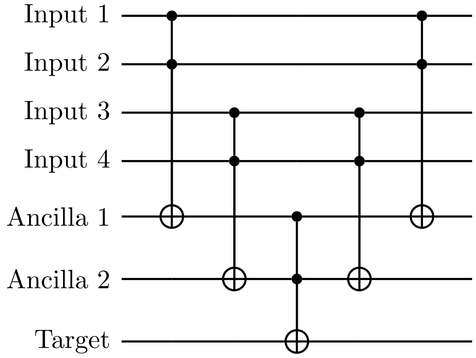
$$CCX\,|101\rangle = |101\rangle$$

$$CCX\,|110\rangle = |111\rangle$$

$$CCX\,|111\rangle = |110\rangle$$

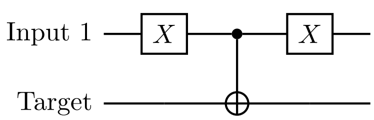
Example:

C^4NOT : Apply NOT to target if 4 control qubits are in state $|1\rangle$?



Example:

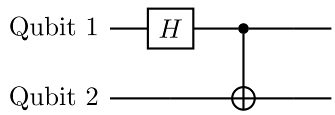
Apply NOT to target if control qubit is on state $|0\rangle$?



5 Basic Quantum Protocols

5.1 Entanglement

Suppose that we have a quantum system in $|\Psi\rangle = |00\rangle$. Let's apply the following operations.



$$\begin{aligned}(H \otimes I)(|00\rangle) &= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\end{aligned}$$

$$CNOT\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\right) = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\Big)$$

If we measure first qubit and observe $|0\rangle$, then the second qubit collapses to $|0\rangle$. Similarly, if we measure the second qubit and observe $|1\rangle$, then the second qubit collapses to $|1\rangle$. This is true even if the qubits are seperated from each other. This is called entanglement.

Bell States

- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
- $|\Phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$
- $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$
- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

The states above are known as the Bell states and they are entangled states. Entangled states can not be written as a tensor product of two subsystems.

5.2 Superdense Coding

Suppose that Alice wants to send Bob two classical bits of information. Alice will send only a single qubit to Bob to achieve this task.

- Initially, Alice and Bob have an entangled pair of qubits in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
- Alice has the first qubit and Bob has the second qubit. Alice wants to send two bits of classical information, ab, to Bob.
- If a=1, Alice applies Z-Gate to her qubit.
- If b=1, Alice applies X and sends to Bob.

ab	Operation	Result
00	-	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$
01	X	$\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$
10	Z	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$
11	XZ	$\frac{1}{\sqrt{2}}\|10\rangle - \frac{1}{\sqrt{2}}\|01\rangle$

- Now Bob has both qubits.
- Bob applies CNOT where first qubit is control and second qubit is target.
- Bob applies Hadamard to first qubit
- He makes the measuerment and observes quantum state $|ab\rangle$

1'st Case - $|00\rangle$

$$
\begin{aligned}
|\Psi\rangle &= CNOT\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}
$$

$$
\begin{aligned}
(H \otimes I)|\Psi\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle = |00\rangle
\end{aligned}
$$

2'nd Case - $|01\rangle$

$$
\begin{aligned}
|\Psi\rangle &= CNOT\left(\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle\right) \\
&= \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|01\rangle
\end{aligned}
$$

$$
\begin{aligned}
(H \otimes I)|\Psi\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)|1\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|1\rangle \\
&= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle = |01\rangle
\end{aligned}
$$