Lecture 3 - Quantum Systems

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2.4.2 Basics, Superposition, Vector Representation

Qubit: Qubit is the smallest unit of quantum information. Any particle with 2 distinct states can act as a qubit. Polarization of a photon (horizontal or vertical), spin of an electron (up or down) can encode a qubit. The two states for a qubit is represented by $|0\rangle$ and $|1\rangle$.

Vector Representation

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superposition

A qubit can be in a linear combination of these states. State of a quantum system is described as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α, β are complex numbers named *amplitudes* satisfying $|\alpha|^2 + |\beta|^2 = 1$. Hence, any vector

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $|\alpha|^2 + |\beta|^2 = 1$ represents a quantum state. Such a vector is called *unit vector* satisfying $\langle\psi|\psi\rangle = 1$.

State Space Postulate: The state of a system is described by a unit vector in a Hilbert space.

Bloch Sphere

Q: How can we represent the quantum states visually?
A: Any quantum state is given by

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

where c_0 and c_1 are complex numbers such that

$$c_0 = r_0 e^{i\phi_0} \quad c_1 = r_1 e^{i\phi_1} \text{ and } |c_0|^2 + |c_1|^2 = 1.$$

$$\begin{aligned} |r_0 e^{i\phi_0}|^2 + |r_1 e^{i\phi_1}|^2 &= 1 \\ r_0 e^{i\phi_0} r_0 e^{-i\phi_0} + r_1 e^{i\phi_1} r_1 e^{-i\phi_1} &= 1 \\ r_0^2 + r_1^2 &= 1 \end{aligned}$$

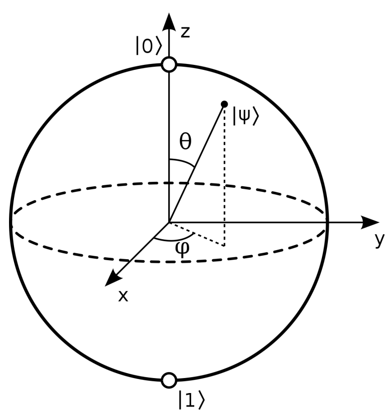
Let $r_0 = cos\theta$ and $r_1 = sin\theta$ for some θ. Replacing r_0 and r_1, we get

$$\begin{aligned} |\psi\rangle &= cos\theta e^{i\phi_0}|0\rangle + sin\theta e^{i\phi_1}|1\rangle \\ &= e^{i\phi_0}(cos\theta|0\rangle + sin\theta e^{i(\phi_1 - \phi_0)}|1\rangle) \end{aligned}$$

Note that $e^{i\phi_0}$ has no observable effect. Let $\phi = \phi_1 - \phi_0$.

$$|\psi\rangle = \cos\theta|0\rangle + \sin\theta e^{i\phi}|1\rangle$$

This formulation allows us to represent any quantum state on the surface of a unit sphere. This representation is known as Bloch Sphere. We choose the range of θ and ϕ so that we have no repetitions. Visit `http://akyrillidis.github.io/notes/quant_post_7` for more details.



$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|1\rangle$$
$$\phi \in [0, 2\pi)$$
$$\theta \in [0, \pi)$$

Note that when $\theta = 0$ we get state $|0\rangle$ and when $\theta = \pi$ we get state $|1\rangle$. States $|0\rangle$ and $-|0\rangle$ have the same representation on the Bloch Sphere.

2.4.3 Evolution of Quantum Systems

Evolution of a Closed System: Evolution of the state vector of a quantum system is linear. It should preserve the norm of the vector.

Q: Which operators can we apply on quantum systems?
A: Let v be a vector representing a quantum state such that $\|v\| = 1$. Let U be any operator acting on v. It should satisfy $\|Uv\| = 1$. Those operators should be also reversible due to nature of quantum mechanics.

The reversible linear operators that preserve the norm of such vectors are *unitary*. An operator U is unitary if $UU^\dagger = U^\dagger U = I$.

Evolution Postulate The evolution of a closed quantum system is described by a unitary transformation.

Unitary operators preserve norms:

$$\|Uv\|^2 = \langle Uv, Uv\rangle = (Uv)^\dagger(Uv) = v^\dagger U^\dagger Uv = v^\dagger v = \|v\|^2$$

A unitary operator U acting on a single qubit is described as a *1-qubit gate*. Operators on the 2-D Hilbert space of a single qubit can be represented as 2x2 matrices.

Example: NOT Operator

$\text{NOT}|0\rangle = |1\rangle \quad \text{NOT}|1\rangle = |0\rangle$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Probabilistic vs. Quantum

Probabistic	Quantum
$a[0] + b[1]$	$\alpha\|0\rangle + \beta\|1\rangle$
$a + b = 1,\ a.b \geq 0$	$\|\alpha\|^2 + \|\beta\|^2 = 1$
$a, b \in \mathbb{R}$	$\alpha, \beta \in C$
stochastic vectors	unit vectors
stochastic operators	unitary operators
$(\|\| \quad \|\|)_1$	$(\|\| \quad \|\|)_2$

2.4.4 Quantum Systems with Multiple Qubits

Composite Systems Postulate: When 2 physical systems are treated as one combined system, the state space of the combined physical system is the tensor product $H_1 \otimes H_2$ of the state spaces H_1, H_2 of the subsystems.

$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1 \psi_2\rangle$

Vector Representation of Basis States for 2 Qubits

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

How to obtain the vector representations?

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In general, state of a two-qubit system is given by

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$ and represented with the following vector: $\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$.

Example

If the first qubit is in state $|q_1\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$ and the second qubit in state $|q_2\rangle = \frac{-1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, then the state of the combined system is

$q_1 \otimes q_2 = \frac{-1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{-1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$.

Now let's apply NOT operator only to first qubit in the previous example. Then we obtain the following quantum state:

$$\frac{-1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle + \frac{-1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle$$

Q: What is the operator that corresponds to this transform?

$$U \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

A: Applying NOT operator only to first qubit is equivalent to applying NOT to the first and IDENTITY to the second qubit. Therefore the corresponding operator is:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Qubits

When there are n qubits, then the basis states are of the form $|a_1, \ldots, a_n\rangle$' where $a_i \in \{0, 1\}$. Then the state space of the system is $H_1 \otimes H_2 \otimes \cdots \otimes H_n$, 2^n-dimensional Hilbert space.

In general, quantum state of an n-qubit system is given by

$$\sum_{i=0}^{2^n-1} \alpha_i |i\rangle \quad \text{where} \quad \alpha_i \in \mathbb{C} \quad \text{and} \sum |\alpha_i|^2 = 1.$$

$\{|i\rangle\}$ is the basis where i is written as n-bit binary number.

2.4.5 Measurement

When some properties of the system is measured, then it is no longer closed and evolution postulate no longer works.

Measurement Postulate For a given orthonormal basis $B = \{|\phi_i\rangle\}$ of a state space H_A for a system A, it is possible to perform Von Neumann measurement with respect to basis B such that given a state $|\psi\rangle = \sum \alpha_i |\phi_i\rangle$, outputs a label i with probability $|\alpha_i|^2$ and leaves the system in $|\phi_i\rangle$.

Example:

If the qubit is in state $\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$ and measurement basis is $\{|0\rangle, |1\rangle\}$ then

$$P(|0\rangle) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \qquad P(|1\rangle) = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = \frac{2}{3}$$

Note that for $|\psi\rangle = \sum \alpha_i |\phi_i\rangle$, $\alpha_i = \langle \phi_i | \psi \rangle$. Another way to obtain $P(|0\rangle) = \alpha_0^2$ is the following:

$$\alpha_0 = \langle 0|\psi\rangle = \langle 0| \left(\tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) = \tfrac{1}{\sqrt{3}}\langle 0|0\rangle + \tfrac{2}{\sqrt{3}}\langle 0|1\rangle = \tfrac{1}{\sqrt{3}}.1 + \tfrac{2}{\sqrt{3}}.0 = \tfrac{1}{\sqrt{3}}$$

There are different types of measurement than we discuss here. Please consult the book and the extra resources in the course website. Here is an idea how we can define measurement operators.

$$P(i) = |\alpha_i|^2 = \alpha_i^* \alpha_i = \langle \psi | \phi_i \rangle \langle \phi_i | \psi \rangle = \langle \psi | \phi_i^\dagger \phi_i | \psi \rangle$$

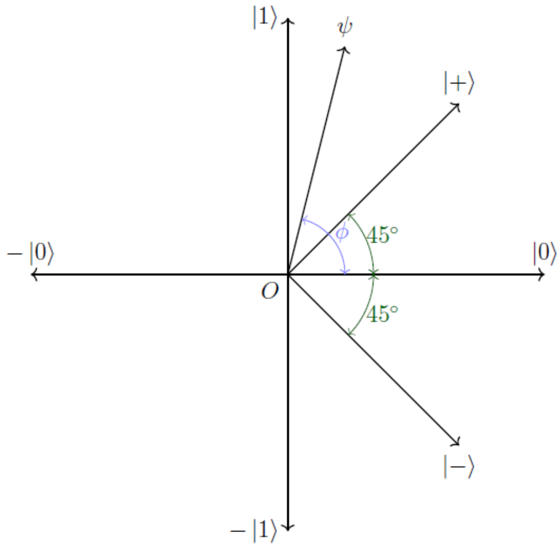
Note that for states $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ the statistics of any measurement is the same.

Measurement in a different basis

Let $\beta = \{|+\rangle, |-\rangle\}$ be the basis where

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \text{ and } |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

Let's measure $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ in basis β.



$$\alpha_+ = \langle + | \psi \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\alpha_- = \langle - | \psi \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$P(|+\rangle) = \frac{2+\sqrt{3}}{4} \quad P(|-\rangle) = \frac{2-\sqrt{3}}{4}$$

Example

Suppose that we measure both qubits of a composite system in state

$$|\psi\rangle = \sqrt{\frac{1}{11}}|00\rangle + \sqrt{\frac{5}{11}}|01\rangle + \sqrt{\frac{2}{11}}|10\rangle + \sqrt{\frac{3}{11}}|11\rangle.$$

The probability of observing each basis state is given by:

$$P(|00\rangle) = \frac{1}{11}, \quad P(|01\rangle) = \frac{5}{11}, \quad P(|10\rangle) = \frac{2}{11}, \quad P(|11\rangle) = \frac{3}{11}$$

Q: What if we measure only the first qubit?
A: The probability observing $|0\rangle$ in this case is $P(|00\rangle) + P(|01\rangle) = \frac{6}{11}$. What happens to the other qubit? Let's rewrite our quantum state as follows:

$$|\psi\rangle = \sqrt{\frac{6}{11}}|0\rangle \otimes (\frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle) + \sqrt{\frac{5}{11}}|1\rangle \otimes (\sqrt{\frac{2}{5}}|0\rangle + \sqrt{\frac{3}{5}}|1\rangle)$$

We observe $|0\rangle$ with probability $\frac{6}{11}$ in which case the second qubit is left in state $\frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle$.