FE 587, Option Pricing via Monte Carlo

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1 Background

Here, we focus on the fundamental essentials of the theory as a first-order approximation to reality.

Suppose we have an asset (e.g. stock) with a price process S_t. We have also a riskless asset (the bank account or the government bond, denoted by B).

An option is a financial instrument giving one the right but not the obligation to make a specified transaction at (or by) a specified date at a specified price.

Call options give one the right to buy. Put options give one the right to sell.

- European options
  give one the right to buy/sell on the specified date, the expiry date, when the option expires or matures.
- American options
  give one the right to buy/sell at any time prior to or at expiry. Asian options depend on the average price over a period.
- Lookback options
  depend on the maximum or minimum price over a period
- Barrier options depend on some price level being attained

The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the exercise price or the strike price K.

The payoff is the value at expiry. If the asset is of price S_T and the strike price is K, a European call option is worth $C_T = (S_T - K)^+$. For example, consider a call option with strike price $K = 80$ at time T. If the stock at time T is 100 USD and we have a right to buy for 80, the option is worth 20 USD. Similarly,

1.1 The Bond, the riskless bank account

Fixed compound interest rate for borrowing and lending r. If you have R_0 USD now, you will have at t

$$R_t = R_0 \exp(rt)$$

1.2 The PutCall Parity

This relation is independent of the model that is assumed for the stock-price behaviour. It is a model-independent result based on the no-arbitrage assumption.

Take a portfolio

$$\Pi_t = S_t + P_t + C_t$$

This portfolio is created as follows: at time t

- Buy 1 unit of stock at price S_t
- Buy a put option at strike price K, with expiry time at T

- Write a call option at strike price K, with expiry time at T

We can have to outcomes:

- $S_T \geq K$.
  - The put option at strike price K has no value, (as it gives only the right to sell at a cheaper price)
  - The call option we have written gives the buyer to buy at K, so we loose $S_T - K$.
  - The stock we have bought is now worth S_T

  The conditional net price of the portfolio is

$$\Pi_T = S_T - (S_T - K) = K$$

- $S_T < K$.
  - The put option at strike price K has value $K - S_T$, as we have the right to sell at a higher price
  - The call option we have written gives the buyer to buy at a higher price, so we loose nothing.
  - The stock we have bought is now worth S_T

  The conditional net price of the portfolio is

$$\Pi_T = S_T + K - S_T = K$$

So whatever the outcome, the price of this portfolio at time T will be K; it acts like government bond. So its price at time t must be

$$\Pi_t = \exp(-r(T-t))K$$

otherwise it gives an arbitrage opportunity, a possibility to make money at no risk.

Strategies for exploiting the arbitrage opportunity:

- Portfolio is being sold cheaper $\Pi_t < \exp(-r(T-t))K$
  - At t: Lend money $\exp(-r(T-t))K$ from the bank, Buy the portfolio
  - At T: Pay back $\exp(-r(T-t))K\exp(r(T-t)) = K$ to the bank. Enjoy your $\exp(-r(T-t))K - \Pi_t$.
- Portfolio is being sold overpriced $\Pi_t > \exp(-r(T-t))K$
  - At t: Buy the negative of the portfolio:

2 Option Pricing via Monte Carlo

The model for a risk neutral evolution is

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Here, S_t is the price at time t. We also assume that there is the riskless bond with compound interest rate r. W_t is a realisation from the Brownian process and σ^2 is the (constant) volatility.

The solution of this SDE is

$$\log S_t = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

So at strike time, the price will be

$$\begin{aligned} S_T &= S_0 \exp(\left(r - \frac{1}{2}\sigma^2\right) T + \sigma W_T) \\ W_T &\sim \mathcal{N}(0; T) \end{aligned}$$

The payoff for call options is

$$C_T = (S_T - K)^+$$

The payoff for put options is

$$P_T = (K - S_T)^+$$

The price of the call option at $t = 0$ will be

$$e^{-rT} \langle C_T \rangle$$