3.3 MATHEMATICAL INDUCTION

From modus ponens:

$$\begin{array}{cl} p & \text{basis assertion} \\ p \to q & \text{conditional assertion} \\ \hline q & \text{conclusion} \end{array}$$

we can easily derive "double modus ponens":

$$\begin{array}{cl} p_0 & \text{basis assertion} \\ p_0 \to p_1 & \text{conditional assertion} \\ p_1 \to p_2 & \text{conditional assertion} \\ \hline p_2 & \text{conclusion} \end{array}$$

We might also derive triple modus ponens, quadruple modus ponens, and so on. Thus, we have no trouble proving assertions about arbitrarily large integers. For instance,

The initial domino falls.
If any of the first 999 dominoes falls,
then so does its successor.

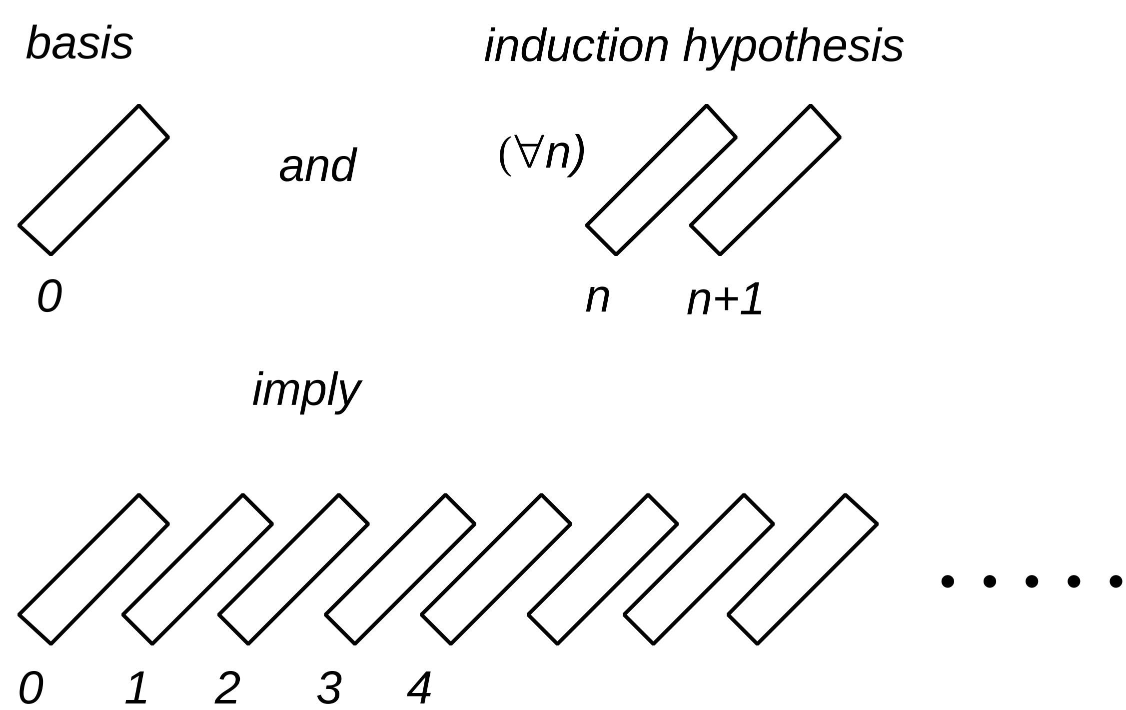
Therefore, the first 1000 dominoes all fall down.

The induction axiom for the integers may be characterized as

THE GREAT LEAP TO INFINITY

Given a countably infinite row of dominoes, suppose that:
(1) The initial domino falls.
(2) If domino n, then so domino $n+1$.

Conclusion: All the dominoes all fall down.



Example 3.3.1: a proof by induction
Calculate the sum of the first k odd numbers:

$$1 + 3 + 5 + \cdots + (2k - 1)$$

Practical Method for General Problem Solving.
Special Case: Deriving a Formula
Step 1. Calculate the result for some small cases.
Step 2. Guess a formula to match all those cases.
Step 3. Verify your guess in the general case.

Step 1. examine small cases

$$\begin{aligned} \text{(empty sum)} &= 0 \\ 1 &= 1 \\ 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \end{aligned}$$

Step 2. It sure looks like $1+3+...+(2k-1) = k^2$.

Step 3. Try to prove this assertion by induction.

$$(\forall k) \left[\sum_{j=1}^{k} (2j - 1) = k^2 \right]$$

(see next page for proof)

Basis Step. $\left[\sum_{j=1}^{k}(2j-1)=k^2\right]$ when $k=0$

Ind Hyp. $\left[\sum_{j=1}^{k}(2j-1)=k^2\right]$ when $k=n$

Ind Step. Consider the case $k=n+1$.

$$\begin{aligned}\sum_{j=1}^{n+1}(2j-1) &= \sum_{j=1}^{n}(2j-1)+[2(n+1)-1]\\ &= \sum_{j=1}^{n}(2j-1)+2n+1\\ &= n^2+2n+1 \text{ by ind. hyp.}\\ &= (n+1)^2 \text{ by factoring } \diamondsuit\end{aligned}$$

Why is induction important to CS majors?

It is the method used to prove that a loop or a recursively defined function correctly calculates the intended result. (just for a start)

Example 3.3.2: another proof by induction

Calculate the sum of the first k numbers:

$$1 + 2 + 3 + \cdots + k$$

Step 1. examine small cases

$$\begin{aligned} \text{(empty sum)} = 0 &= \frac{0 \cdot 1}{2} \\ 1 = 1 &= \frac{1 \cdot 2}{2} \\ 1 + 2 = 3 &= \frac{2 \cdot 3}{2} \\ 1 + 2 + 3 = 6 &= \frac{3 \cdot 4}{2} \\ 1 + 2 + 3 + 4 = 10 &= \frac{4 \cdot 5}{2} \end{aligned}$$

Step 2. Infer pattern: $\displaystyle\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$.

Step 3. Use induction proof to verify pattern.

See next page.

Proposition 3.3.1. $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$.

Basis Step. $\sum_{j=1}^{k} j = \frac{k(k+1)}{2} = \frac{0 \cdot 1}{2}$ when $k = 0$.

Ind Hyp. $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$ when $k = n$.

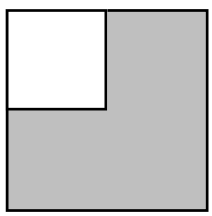
Ind. Step.

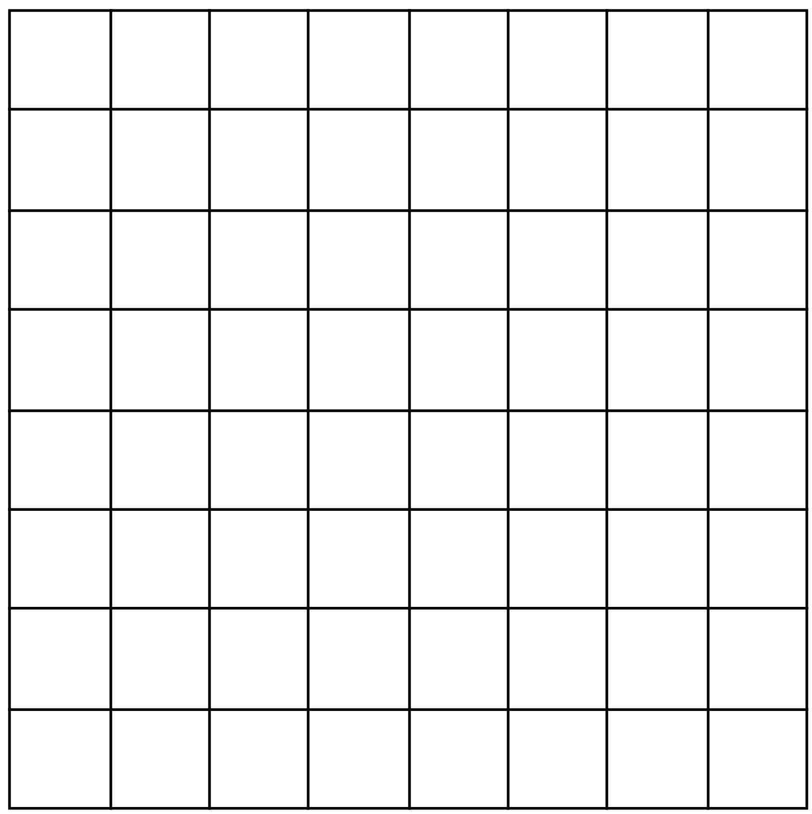
$$
\begin{aligned}
\sum_{j=1}^{n+1} j &= \sum_{j=1}^{n} j + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) && \text{by ind hyp} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} && \text{by arithmetic} \\
&= \frac{n(n+1) + 2(n+1)}{2} && \text{by arithmetic} \\
&= \frac{(n+2)(n+1)}{2} && \text{distrib in numerator} \\
&= \frac{(n+1)(n+2)}{2} && \text{commutativity } \diamondsuit
\end{aligned}
$$

NONALGEBRAIC APPLICATIONS of INDUCTION

Consider tiling a 2^k-by-2^k chessboard.
(k = 3 in the figure below)

with L-shaped tiles, so that one corner-square is left uncovered.





Basis Step. You can do this when k = 0.

Ind Hyp. Assume you can do this for k = n.

Ind. Step. Prove you can do it for k = n+1.

ALTERNATIVE FORMS of INDUCTION

In a proof by induction, verifying the inductive premise means you show that the antecedent of the quantified statement implies the conclusion.

DEF: In a proof by mathematical induction, the ***inductive hypothesis*** is the antecedent of the inductive premise.

Standard 0-based inductive rule of inference:

$$\begin{array}{cl} 0 \in S & \text{basis premise} \\ (\forall n)[n \geq 0 \wedge n \in S \Rightarrow n+1 \in S] & \text{ind prem} \\ \hline (\forall n)[n \geq 0 \Rightarrow n \in S] & \text{conclusion} \end{array}$$

Alternative Form 1. Using an integer other than zero as a basis.

$$\begin{array}{cl} b \in S & \text{basis premise} \\ (\forall n)[n \geq b \wedge n \in S \Rightarrow n+1 \in S] & \text{ind prem} \\ \hline (\forall n)[n \geq b \Rightarrow n \in S] & \text{conclusion} \end{array}$$

Example 3.3.3: using 5 as the basis

$$n^2 > 2n + 1 \text{ for all } n \geq 5$$

Basis Step. $5^2 > 2 \cdot 5 + 1$

Ind Hyp. Assume $k^2 > 2k + 1$ for $k \geq 5$.

Ind. Step.

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \quad \text{by arithmetic} \\
&> (2k+1) + 2k + 1 \quad \text{by ind hyp} \\
&= 4k + 2 \quad \text{by arithmetic} \\
&= 2(k+1) + 2k \quad \text{by arithmetic} \\
&\geq 2(k+1) + 10 \quad \text{since } k \geq 5 \\
&\geq 2(k+1) + 1 \quad \text{since } 10 \geq 1 \ \diamondsuit
\end{aligned}$$

Example 3.3.4: $2^n > n^2$ for all $n \geq 5$.

Basis Step. $2^5 > 5^2$

Ind Hyp. Assume $2^k > k^2$ for $k \geq 5$

Ind. Step.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{arithmetic} \\ &= 2^k + 2^k && \text{arithmetic} \\ &> k^2 + k^2 && \text{ind. hyp.} \\ &> k^2 + (2k+1) && \text{by Example 3.3.3} \\ &= (k+1)^2 && \text{arithmetic } \diamondsuit \end{aligned}$$

Example 3.3.5: Prove that any postage of 8 cents or more can be created from nothing but 3-cent and 5-cent stamps.

Basis Step. $8 = 1 \cdot 3¢ + 1 \cdot 5¢$

Ind Hyp. Assume n¢ possible from 3's and 5's.

Ind. Step. Try to make $(n+1)$¢ postage.

Suppose that $n = r \cdot 3¢ + s \cdot 5¢$

Case 1: $s \geq 1$. Then $n + 1 = \ldots$

Case 2: $s = 0$. Then $n + 1 = \ldots$

Alternative Form 2. Inductive hyposthesis is that the first n dominoes all fall down.

$$\begin{array}{ll} b \in S & \text{basis premise} \\ (\forall n)[n \geq b \wedge (\forall k \leq n)[k \in S] \Rightarrow n+1 \in S] & \text{ind p} \\ \hline (\forall n)[n \geq b \Rightarrow n \in S] & \text{conclusion} \end{array}$$

Example 3.3.6: Prove that every integer $n > 0$ is the product of finitely many primes.

Basis Step. 1 is the empty product.

Ind Hyp. Assume that $1, \ldots, n$ are each a product of finitely many primes.

Ind Step.
(1) Either $n+1$ is prime, or $\exists b, c \in \mathcal{Z}$ such that $n+1 = bc$. (law of excl middle, def of prime)
(2) But b and c are the products of finitely many primes. (by Ind Hyp)
(3) Thus, so is bc. $\diamondsuit$

Mind-Benders re Induction

1. 2/3 ancestry
2. All solid billiard balls are the same color.
3. Everyone is essentially bald.