The 4 Color Theorem

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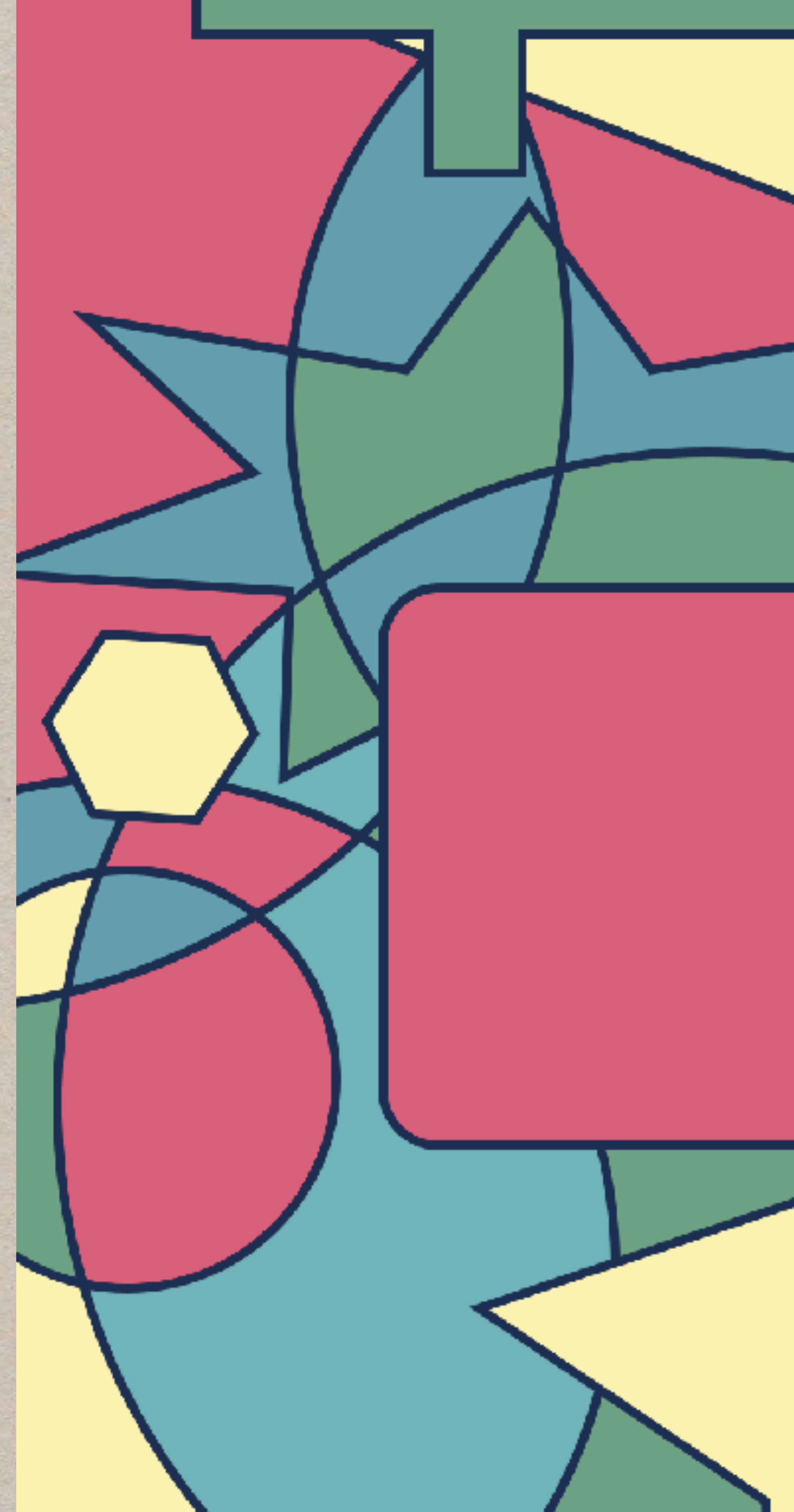
What is it?

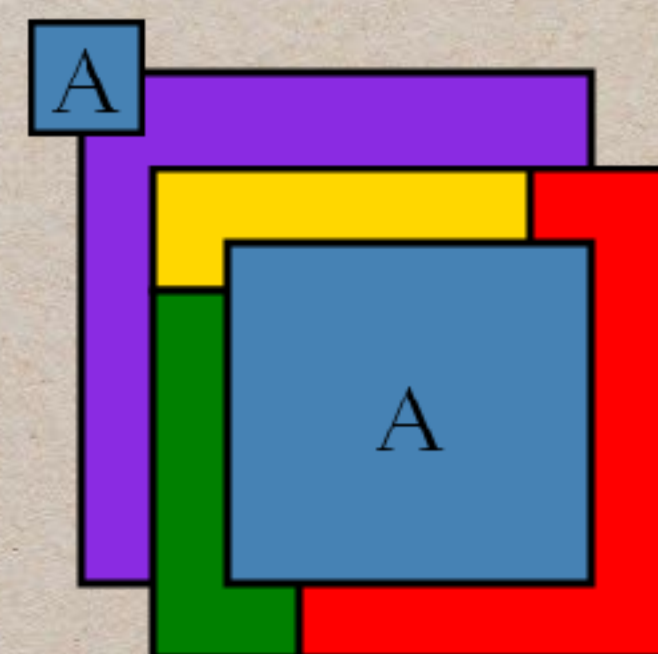
- Given any separation of a plane into contiguous regions producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Regions are adjacent if they share a boundary segment; two regions that share only isolated boundary points are not considered adjacent.

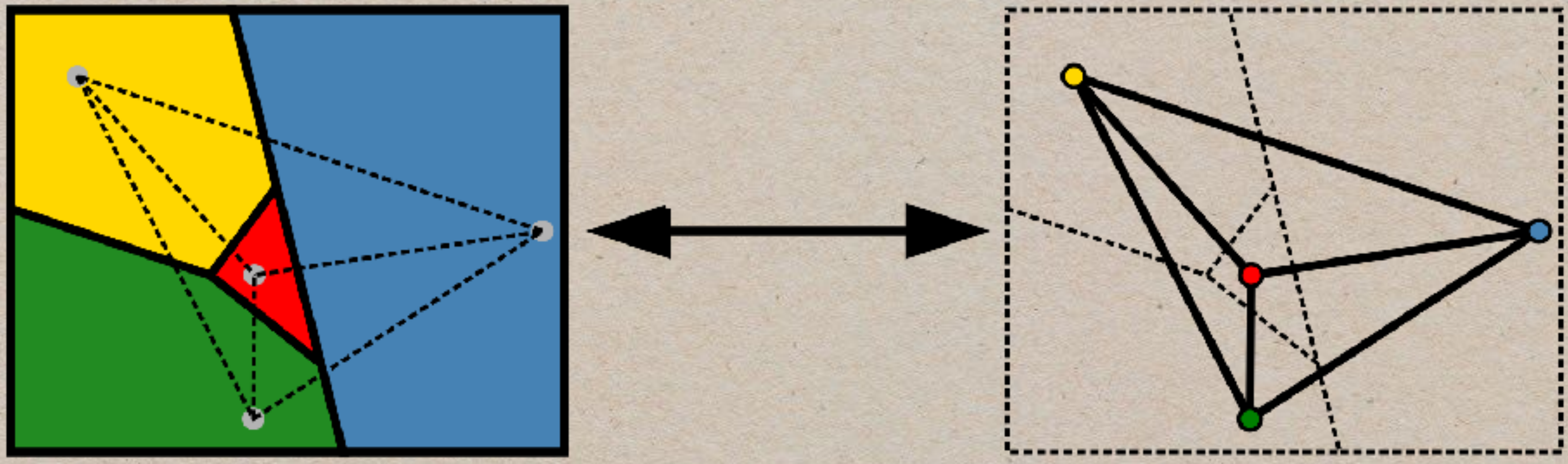
Bizarre regions, such as those with finite area but infinitely long perimeter, are not allowed; maps with such regions can require more than four colors.

If we required the entire territory of a country to receive the same color, then four colors are not always sufficient.



Graphs

The set of regions of a map can be represented more abstractly as an undirected graph that has a vertex for each region and an edge for every pair of regions that share a boundary segment. Vertices of every planar graph can be colored with at most four colors without two adjacent vertices receiving the same color, or, in other words: every planar graph is four colorable.

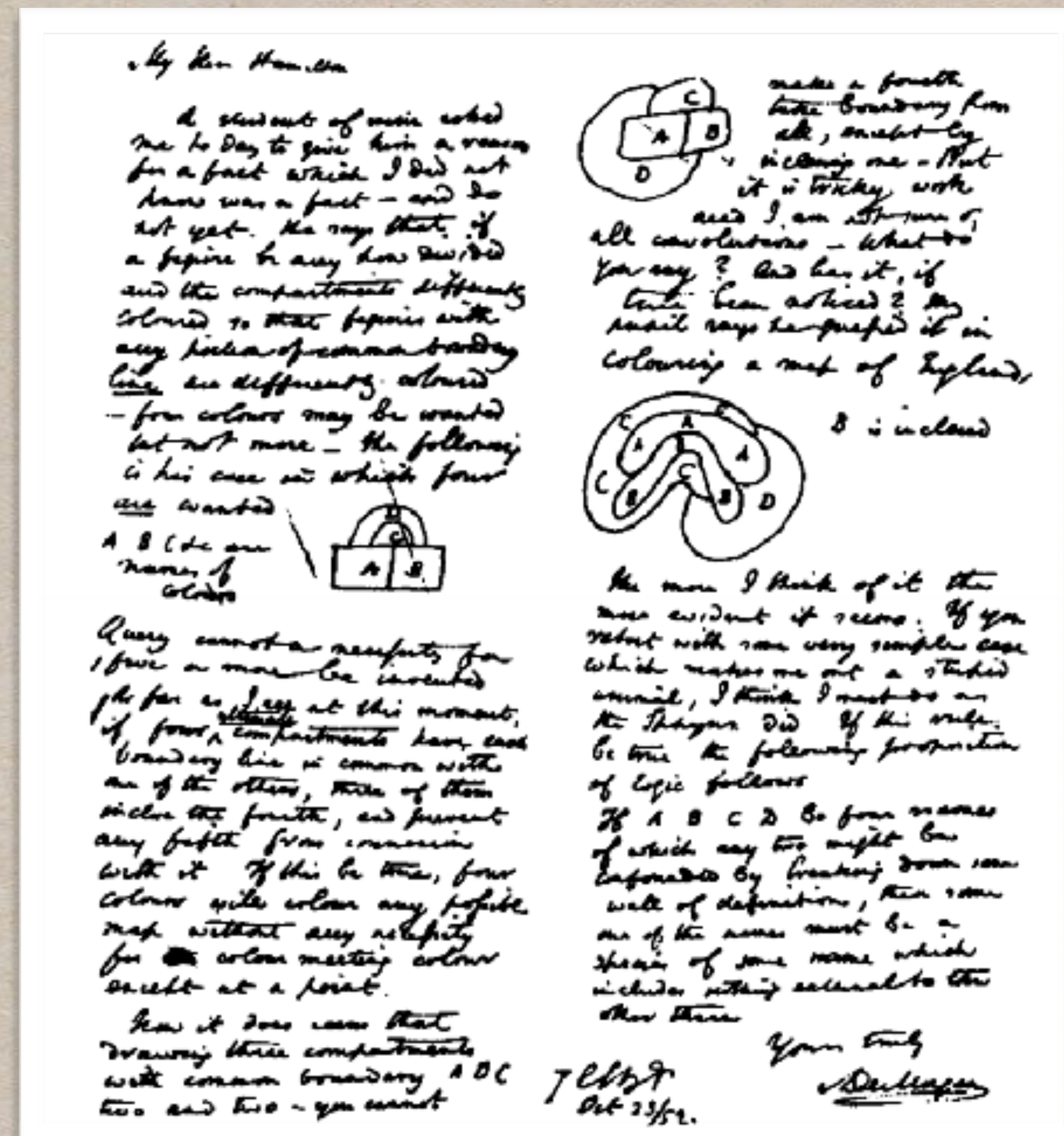


History

The conjecture was first proposed on October 23, 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed.
At the time, Guthrie's brother, Frederick, was a student of Augustus De Morgan at University College London.
De Morgan was unable to give an answer. On 23 October 1852, he wrote the following to William Hamilton, a famous Irish mathematician & astronomer:

"A student of mine [Guthrie] asked me to day to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be any how divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored—four colors may be wanted but not more."

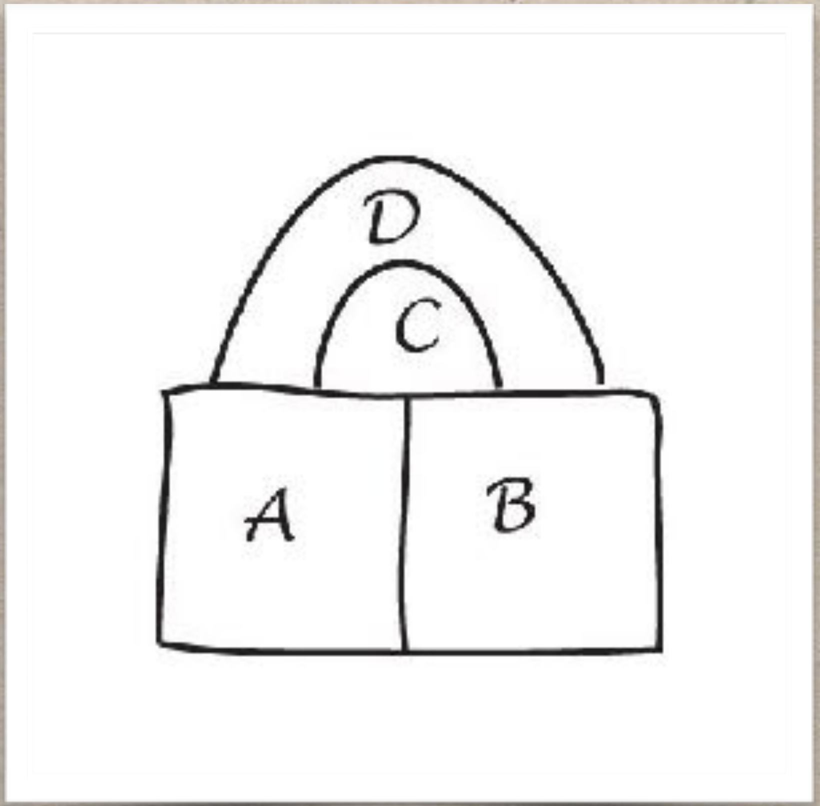
The problem, so simply described, but so difficult to prove, caught the imagination of many mathematicians at the time.

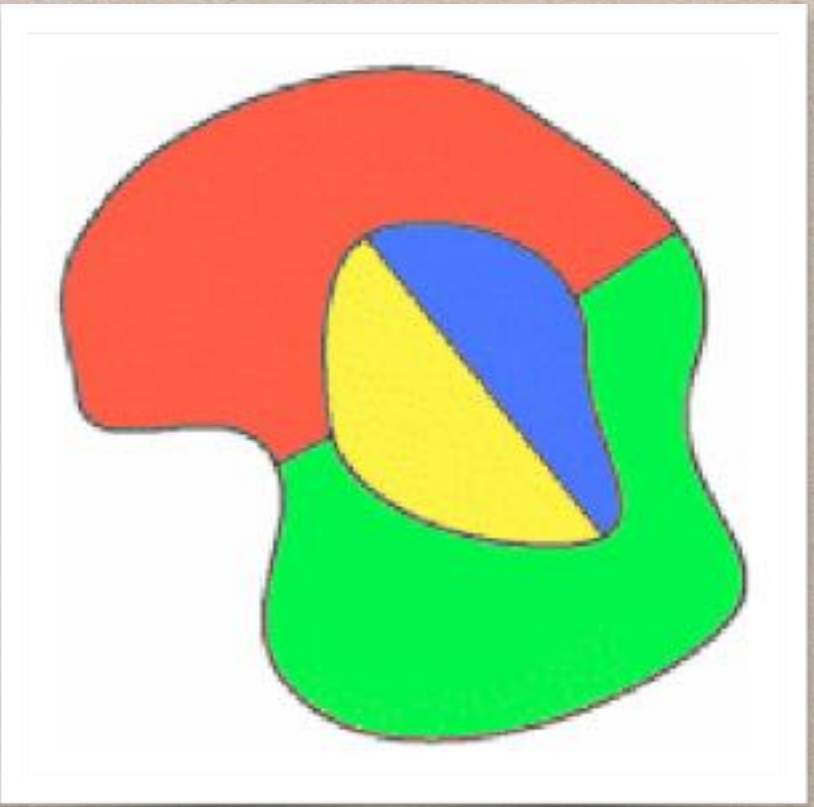


History

There were several early failed attempts at proving the theorem. De Morgan believed that it followed from a simple fact about four regions, though he didn't believe that fact could be derived from more elementary facts.

De Morgan used the fact that in a map with four regions, each touching the other three, one of them is completely enclosed by the others. Now this principle, that four areas cannot each have common boundary with all the other three without inclosure, is not, we fully believe, capable of demonstration upon anything more evident and more elementary; it must stand as a postulate.

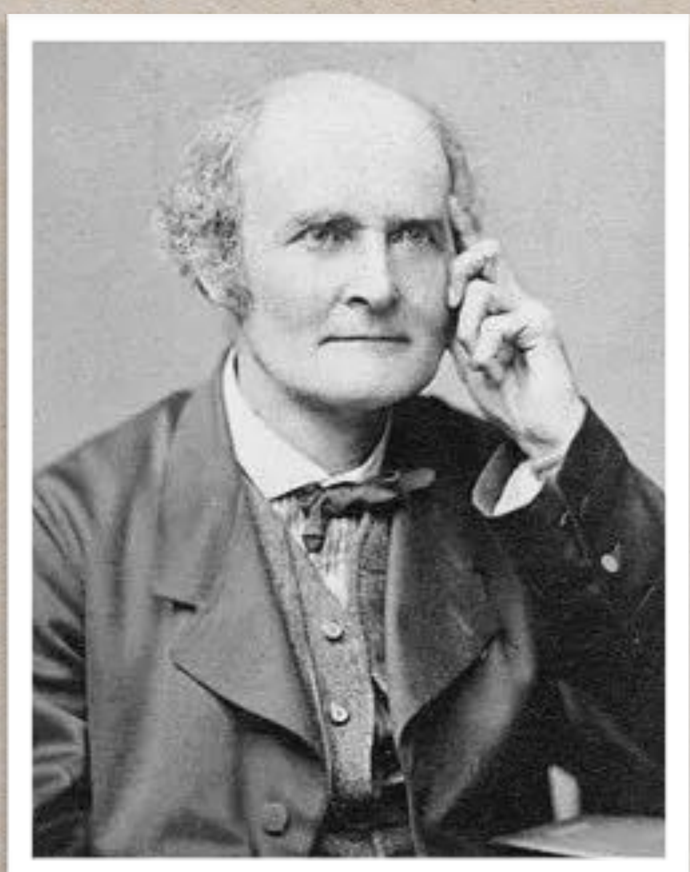


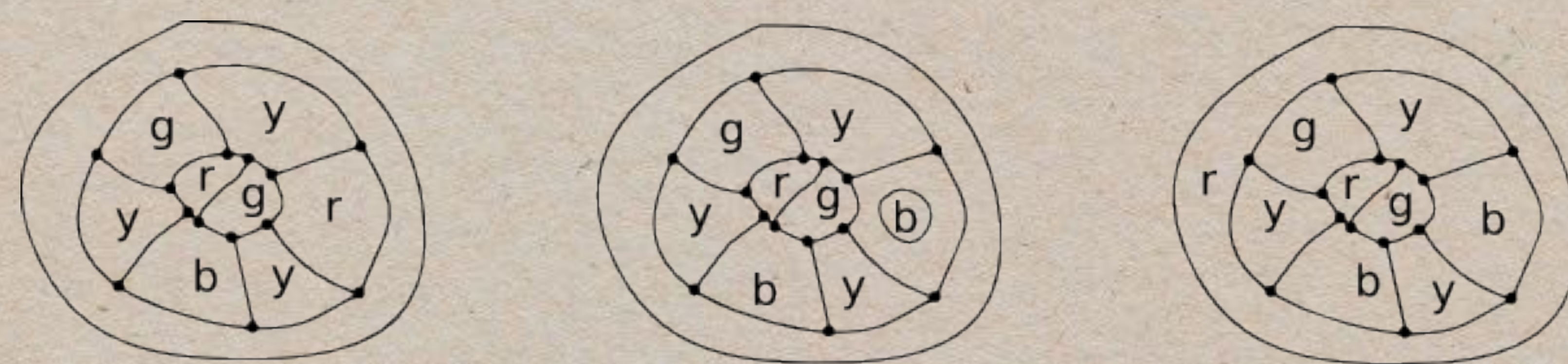


History

In 1878 Arthur Cayley (1821-1895) at a meeting of the London Mathematical Society asked whether anyone had found a solution for De Morgan's original question, but although there had been some interest, no one had made any significant progress.

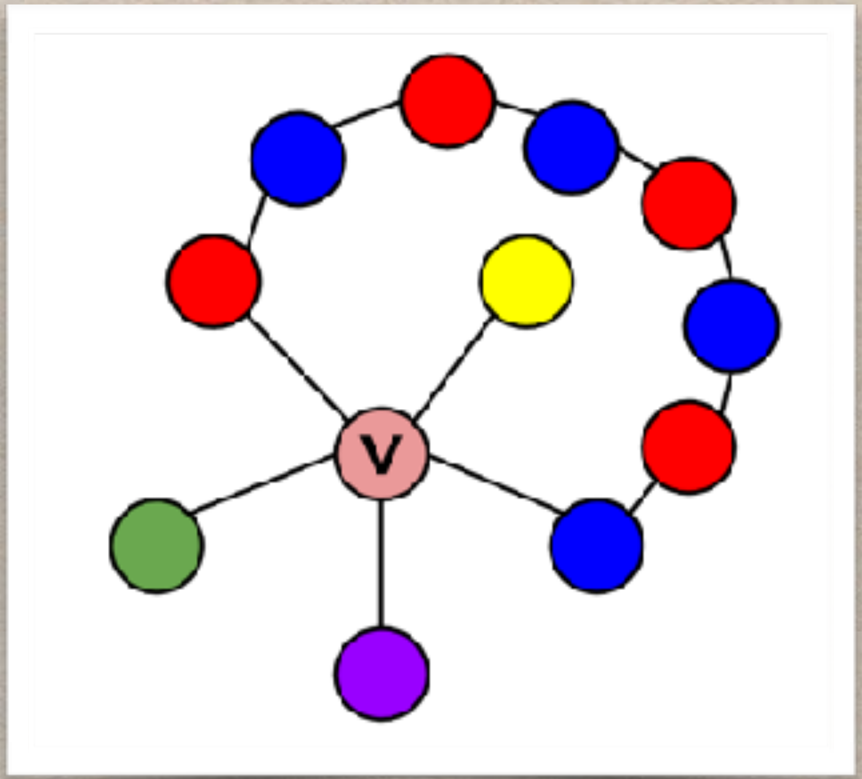
Less than a year later, 1879, Cayley returned with a publication submitted to the Royal Geographical Society (On The Colouring Of Maps). Lacking a proper solution, he nevertheless drastically pushed forth the stagnant search by explaining his shortcomings & attempts at reframing the problem. One of these productive shifts in perspective stemmed from his question: “what if a particular map is already successfully colored with four colors, & we add another area, can we still keep the same coloring?” Essentially an inverse inquiry, it helped start another path towards a general theorem.

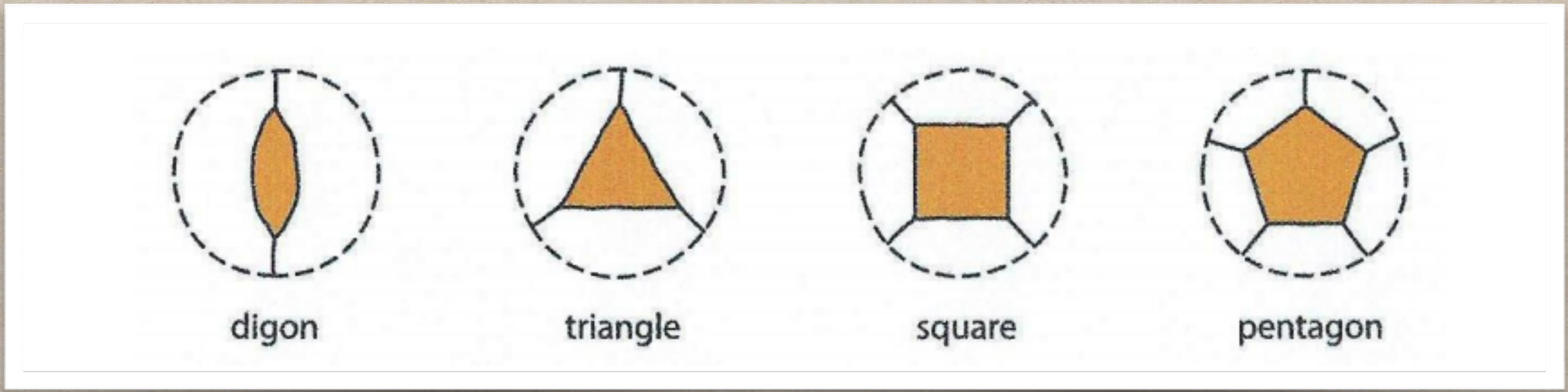


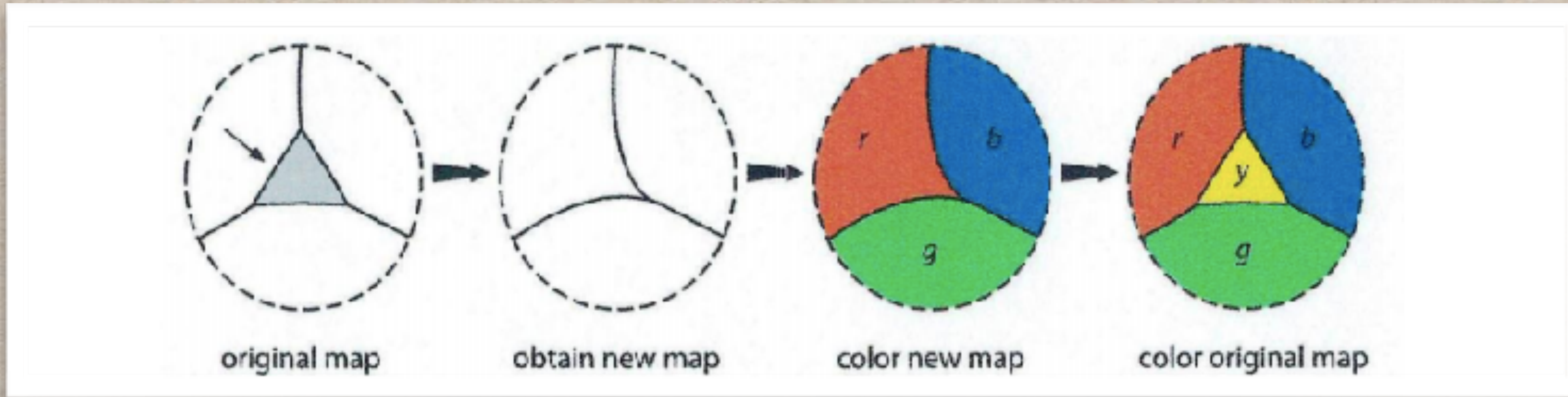


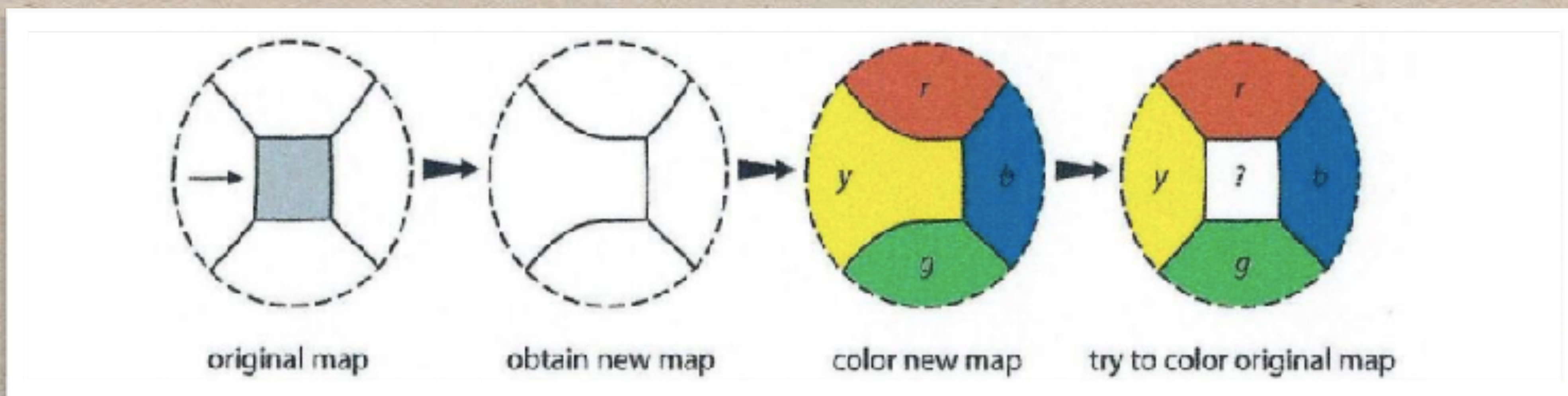
History

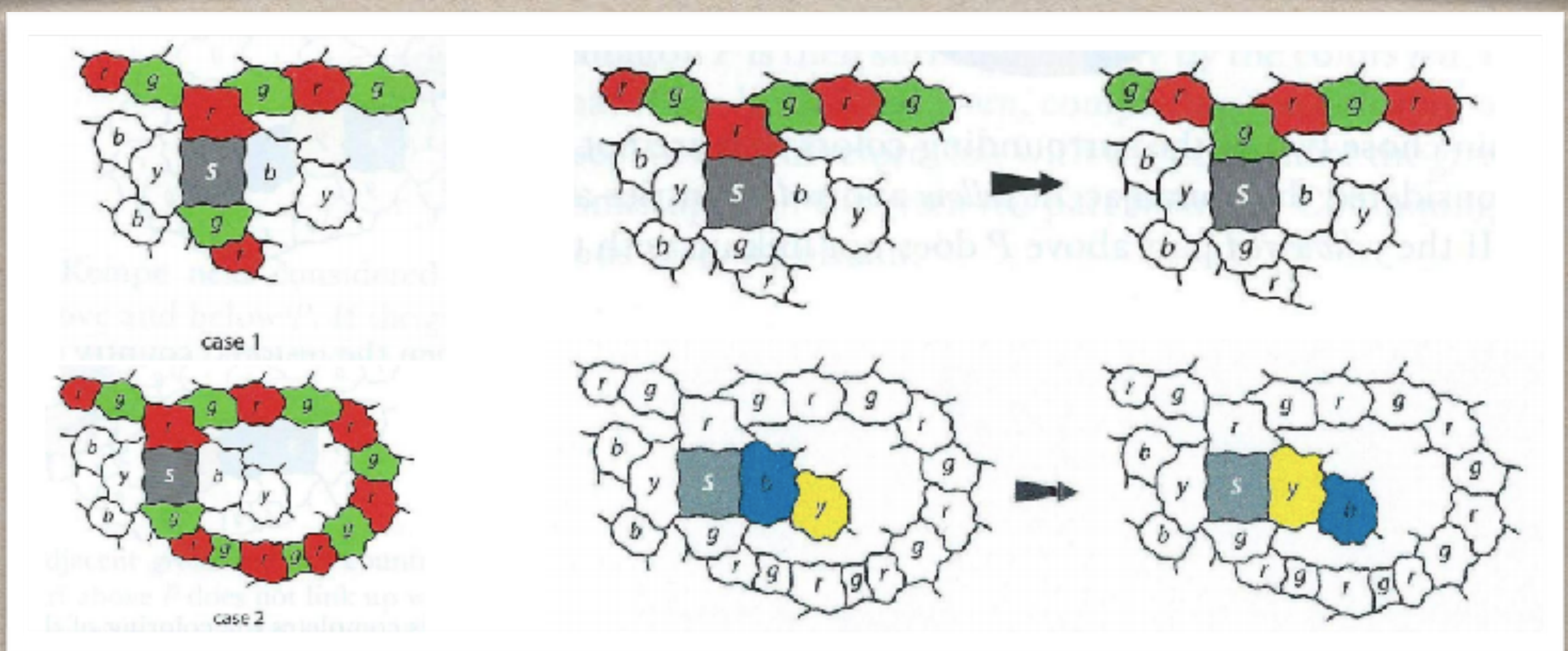
One alleged proof was given by Alfred Kempe in 1879, which was widely acclaimed another was given by Peter Guthrie Tait in 1880. It was not until 1890 that Kempe's proof was shown incorrect by Percy Heawood, and in 1891, Tait's proof was shown incorrect by Julius Petersen—each false proof stood unchallenged for 11 years.

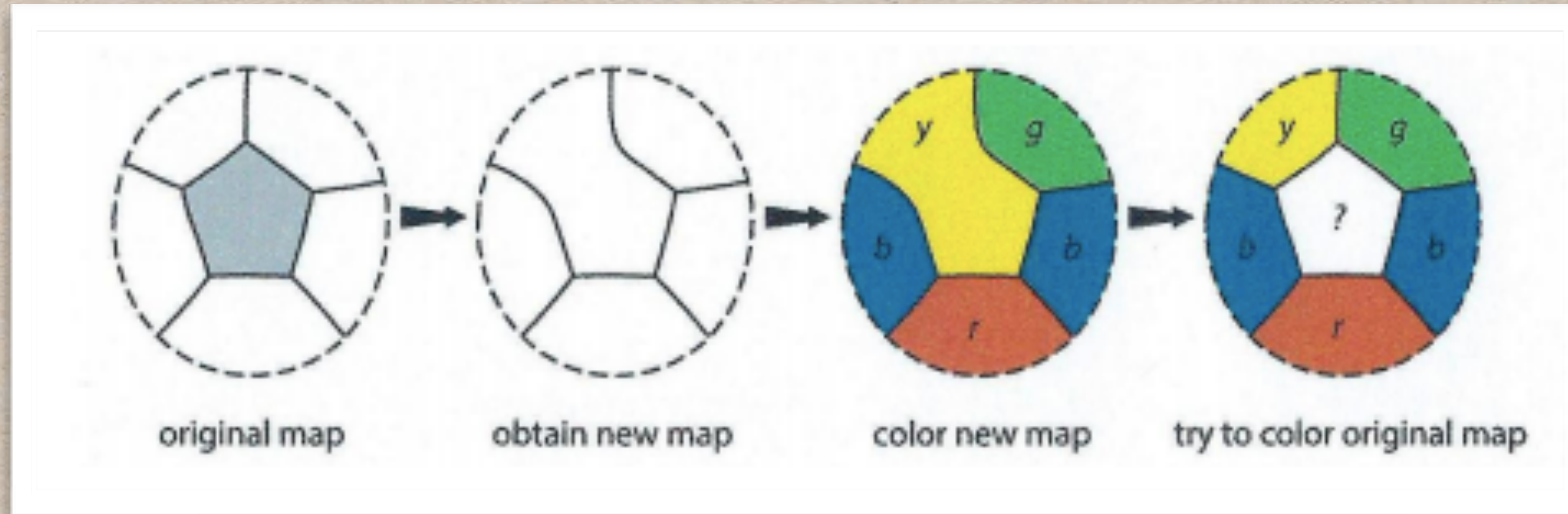


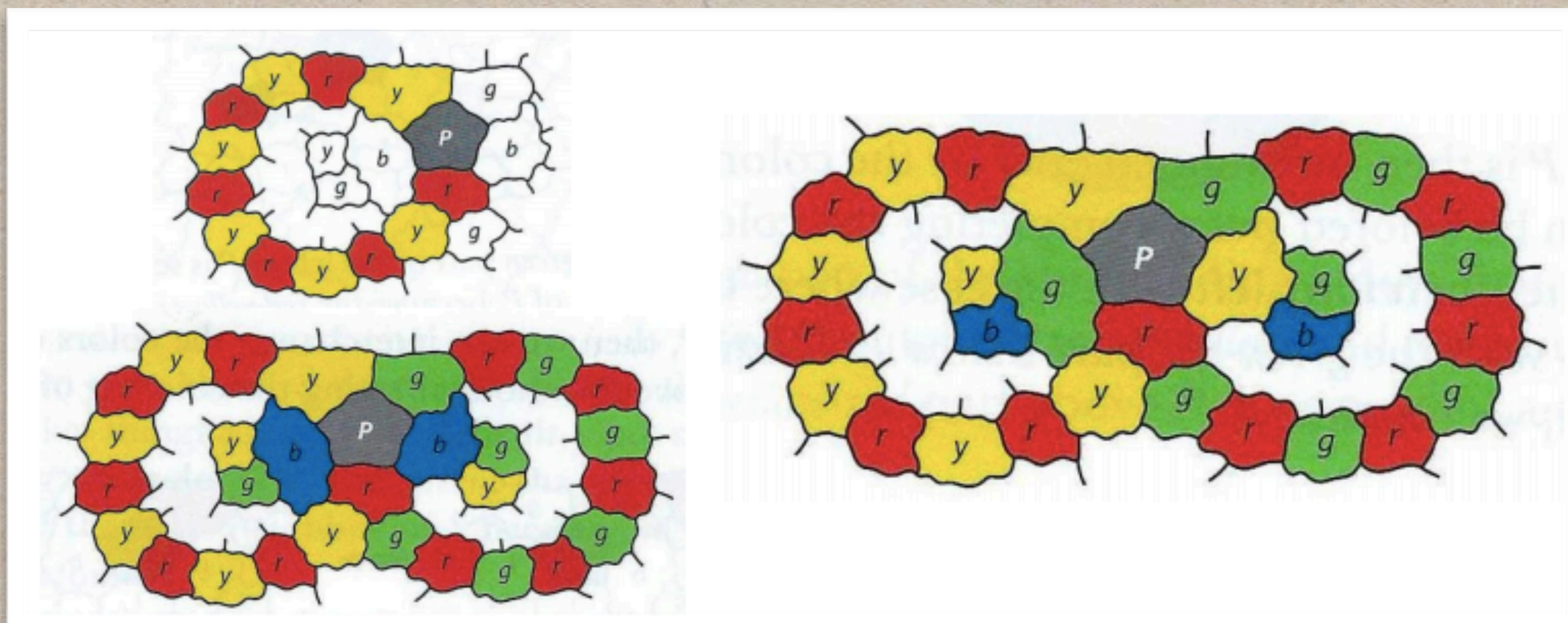



original map
obtain new map
color new map
color original map


original map
obtain new map
color new map
try to color original map


case 1
case 2


original map
obtain new map
color new map
try to color original map



It was not until 1976 that, with the help of modern computers, the 4-color conjecture was finally proven to be true.

Kenneth Appel and Wolfgang Haken at the University of Illinois announced, on June 21, 1976, that they had proved the theorem. They were assisted in some algorithmic work by John A. Koch.

If the four-color conjecture were false, there would be at least one map with the smallest possible number of regions that requires five colors. The proof showed that such a minimal counterexample cannot exist, through the use of two technical concepts:

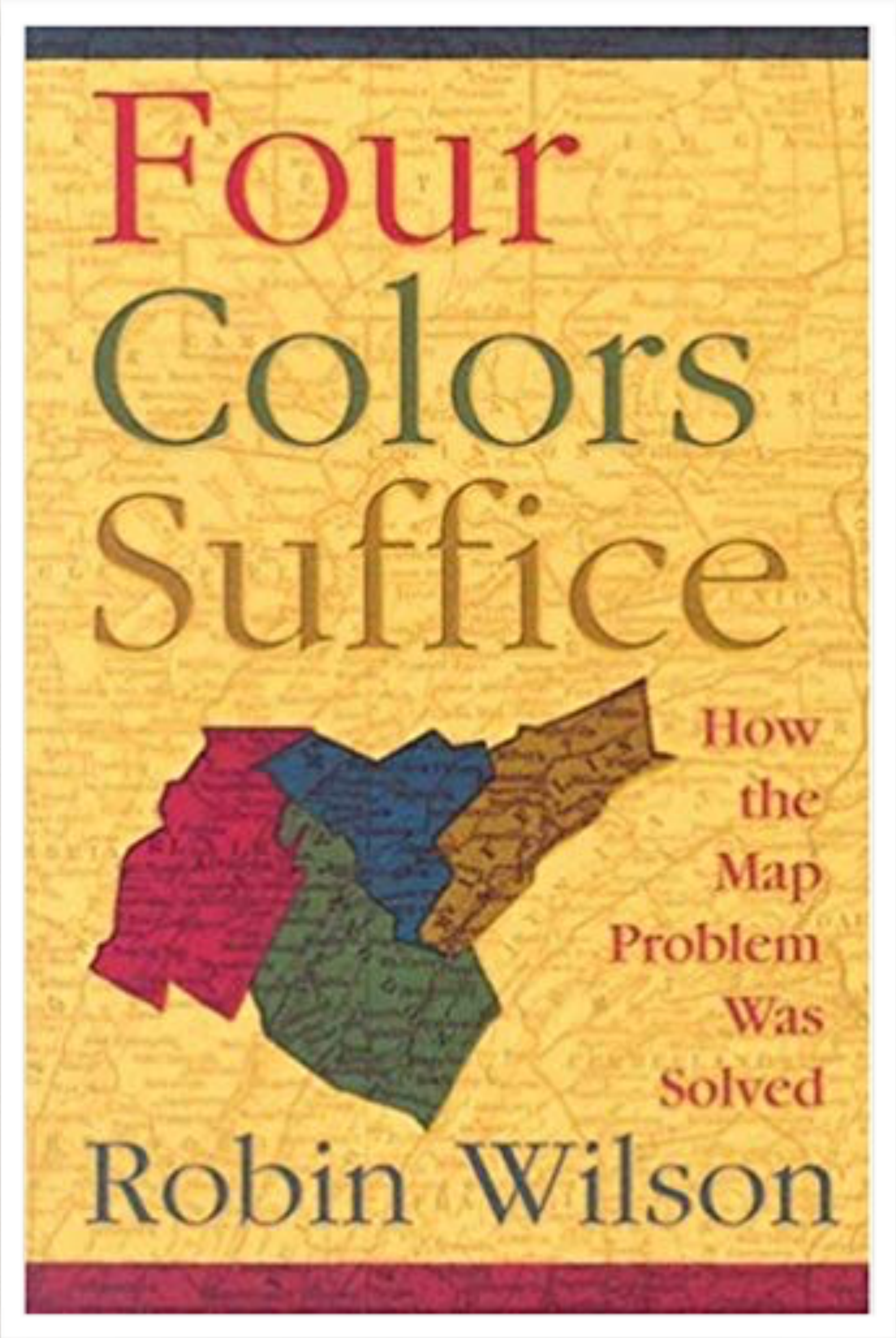
1. **An unavoidable set** is a set of configurations such that every map that satisfies some necessary conditions for being a minimal non-4-colorable
2. **A reducible configuration** is an arrangement of countries that cannot occur in a minimal counterexample.

It was the very first major theorem "proved" through brute-forcing scenarios with a computer.

More on 4 Color Theorem

https://www.mathpages.com/home/kmath266/kmath266.htm

https://www.youtube.com/watch?v=42-ws3bkrKM&t=692s



References

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