What is Real Number?

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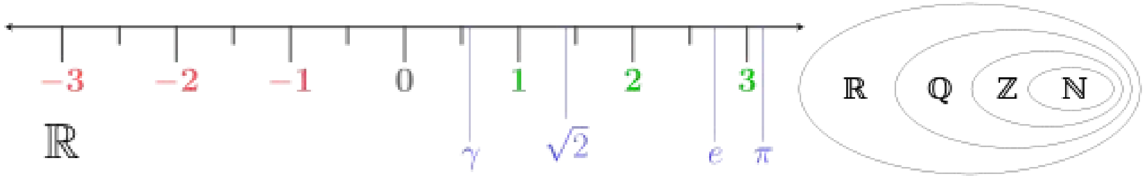
ℝ Basic Definition



->The set of real numbers include all the rational numbers, such as the integer −7 and the fraction 5/3, and all the irrational numbers, such as √2 (1.41421356…), π (3.14159265…)

In the 17th century, Descartes distinguished between real and imaginary roots of polynomials, then he used adjective "*real*" in this context.

Some explicit construction models

1) Construction from Cauchy sequences
2) Construction by Dedekind cuts
3) Stevin's construction
4) Construction using hyperreal numbers
5) Construction from surreal numbers
6) Construction from Z (Eudoxus reals)
7) Other methods

Current Formal Definition

The current standard axiomatic definition is that real numbers form the unique complete totally ordered field (**R** ; + ; x ; <)

The synthetic approach axiomatically defines the real number system as a complete ordered field. This model for the real number system consists of a set R, two binary operations + and × on R (called addition and multiplication, respectively), and a binary relation ≤ on R, satisfying the following properties:

Synthetic approach

Let $\mathbb{R}$ denote the set of all real numbers. Then:

- The set $\mathbb{R}$ is a field, meaning that addition and multiplication are defined and have the usual properties.
- The field $\mathbb{R}$ is ordered, meaning that there is a total order $\geq$ such that, for all real numbers x, y and z:
    - if $x \geq y$ then $x + z \geq y + z$;
    - if $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.
- The order is Dedekind-complete; that is: every non-empty subset S of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound (also called supremum) in $\mathbb{R}$.

The last property is what differentiates the reals from the rationals. For example, the set of rationals with square less than 2 has a rational upper bound (e.g., 1.5) but no rational least upper bound, because the square root of 2 is not rational.

Field Properties

1)(**R**, +, ×) forms a field. In other words,

- For all x, y, and z in **R**, $x + (y + z) = (x + y) + z$ and $x \times (y \times z) = (x \times y) \times z$. (associativity of addition and multiplication)
- For all x and y in **R**, $x + y = y + x$ and $x \times y = y \times x$. (commutativity of addition and multiplication)
- For all x, y, and z in **R**, $x \times (y + z) = (x \times y) + (x \times z)$. (distributivity of multiplication over addition)
- For all x in **R**, $x + 0 = x$. (existence of additive identity)
- 0 is not equal to 1, and for all x in **R**, $x \times 1 = x$. (existence of multiplicative identity)
- For every x in **R**, there exists an element $-x$ in **R**, such that $x + (-x) = 0$. (existence of additive inverses)
- For every $x \neq 0$ in **R**, there exists an element x^{-1} in **R**, such that $x \times x^{-1} = 1$. (existence of multiplicative inverses)

Ordered Set Properties

2) ($\mathbf{R}$, $\leq$) forms a totally ordered set.

- For all x in $\mathbf{R}$, $x \leq x$. (reflexivity)
- For all x and y in $\mathbf{R}$, if $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)
- For all x, y, and z in $\mathbf{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)
- For all x and y in $\mathbf{R}$, $x \leq y$ or $y \leq x$. (totalness)

3)The field operations + and × on **R** are compatible with the order ≤.

- For all x, y and z in **R**, if $x \leq y$, then $x + z \leq y + z$. (preservation of order under addition)
- For all x and y in **R**, if $0 \leq x$ and $0 \leq y$, then $0 \leq x \times y$ (preservation of order under multiplication)

4)The order ≤ is *complete* in the following sense: every non-empty subset of **R** bounded above has a least upper bound. In other words,

- If A is a non-empty subset of **R**, and if A has an upper bound, then A has a least upper bound u, such that for every upper bound v of A, $u \leq v$.

THANK YOU FOR LISTENING!

Sources:

https://en.wikipedia.org/wiki/Construction_of_the_real_numbers#Explicit_constructions_of_models

https://en.wikipedia.org/wiki/Real_number#Axiomatic_approach